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nonlocal conditions

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ABSTRACT

In this paper, we study a class of neutral impulsive functional differential equations with nonlocal conditions. We suppose that the linear part satisfies the Hille–Yosida condition on a Banach space and it is not necessarily densely defined. We give some sufficient conditions ensuring the existence of integral solutions and strict solutions. To illustrate our abstract results, we conclude this work by an example.

1. INTRODUCTION

The study of impulsive functional differential equations is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place "instantaneously" in the form of impulses. The theory of impulsive differential and functional differential equations has been extensively developed; see the monographs of Bainov and Simeonov [6], Lakshmikantham et al. [10], and Samoilenko and Perestyuk [11] where numerous properties of their solutions are studied, and detailed bibliographies are given.

Key words and phrases: Integrated semigroup, Hille–Yosida condition, Integral solution, Strict solution, Nonlocal conditions

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In this paper we study the existence of solutions for semilinear neutral impulsive functional differential equation with nonlocal conditions. More precisely, we consider the following Cauchy problem on a general Banach space X:

$$
(1.1)\qquad \begin{cases} \dfrac{d}{dt}\big[x(t) - F\big(t, x\big(h_1(t)\big)\big)\big] \\ \quad = A\big[x(t) - F\big(t, x\big(h_1(t)\big)\big)\big] + G\big(t, x\big(h_2(t)\big)\big), \quad t \in J,\ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k\big(x\big(t_k^-\big)\big), \quad k = 1, \ldots, m, \\ x(0) + g(x) = x_0 \in X, \end{cases}
$$

where $J = [0, b]$, $x(\cdot)$ takes values in Banach space X with the norm $\|\cdot\|$, $A : D(A) \subseteq X \to X$ is a nondensely defined linear operator and generates an integrated semigroup $\{S(t)\}_{t \geqslant 0}$. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. F, G, g, I_k $(k = 1, 2, \ldots, m)$ and h_1, h_2 are given functions to be specified later.

In the past few years theorems about existence, uniqueness and continuous dependence of impulsive functional differential abstract evolution Cauchy problems with nonlocal conditions have been studied by Akca, Covachev and Al-Zahrani [2], by Fu [8], by Anguraj and Karthikeyan [3] and in the references therein.

In all the work the linear operator A is always defined densely in X and satisfies the Hille–Yosida condition so that it generates a C_0-semigroup or an analytic semigroup. However, as indicated in [7], we sometimes need to deal with the nondensely defined operators. For example, when we look at a one-dimensional heat equation with the Dirichlet condition on [0, 1] and consider $A = \frac{\partial^2}{\partial^2 x}$ in $C([0, 1]; R)$, in order to measure the solution in the sup-norm we take the domain

$$
D(A) = \big\{x \in C^2([0, 1]; R) \mid x(0) = x(1) = 0\big\},
$$

and then it is not dense in $C([0, 1], R)$ with the sup-norm. The example presented in Section 4 also shows the advantages of nondensely defined operators in handling some practical problems. See [7,1] for more examples and remarks concerning the nondensely defined operators.

Up to now there have been very few papers in this direction dealing with the existence of solutions for the nondensely impulsive functional differential equations. Our purpose here is to extend the results of densely defined impulsive functional differential evolution equations with nonlocal conditions to nondensely defined impulsive functional differential evolution equations with nonlocal conditions.

This paper is organized as follows. In Section 2 we introduce some preliminaries about the theory of integrated semigroup. In Section 3 we establish the existence of integral solutions and strict solutions for (1.1). Finally, in Section 4 an example is presented to illustrate the applications of the obtained results.

2. PRELIMINARIES

In this section, we will introduce some preliminaries on the theory of integrated semigroup which are required in this paper.

Definition 2.1 [4]. Let X be a Banach space. An integrated semigroup is a family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators $S(t)$ on X with the following properties:

(i) $S(0)=0$;
(ii) $t\to S(t)$ is strongly continuous;
(iii) $S(t)S(s)=\int_0^t[S(s+\tau)-S(\tau)]\,d\tau$, for all $t,s\geqslant 0$.

Definition 2.2 [4]. An operator A is said to be the generator of an integrated semigroup if there exists $\overline{\omega}\in R$ such that $(\overline{\omega},+\infty)\subset\rho(A)$ and there exists a strongly continuous exponentially bounded family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators such that $S(0)=0$ and $(\lambda I-A)^{-1}=\lambda\int_0^\infty e^{-\lambda t}S(t)\,dt$ exists for all $\lambda>\overline{\omega}$.

Definition 2.3 [9].

(i) An integrated semigroup $\{S(t)\}_{t\geqslant 0}$ is said to be locally Lipschitz continuous, if for all $b>0$, there exists a constant $L>0$ such that $\|S(t)-S(s)\|\leqslant L|t-s|$, $t,s\in[0,b]$.

(ii) An integrated semigroup $\{S(t)\}_{t\geqslant 0}$ is said to be nondegenerate, if $S(t)x=0$ for all $t\geqslant 0$ implies that $x=0$.

Definition 2.4 [9]. We say that a linear operator A satisfies the Hille–Yosida condition (HY) if there exists $\overline{M}\geqslant 1$ and $\overline{\omega}\in R$ such that $(\overline{\omega},+\infty)\subset\rho(A)$ and

$$\text{(HY)}\qquad \sup\left\{(\lambda-\bar{\omega})^n\left\|R(\lambda,A)^n\right\|,n\in N,\lambda>\overline{\omega}\right\}\leqslant\overline{M},$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A)=(\lambda I-A)^{-1}$.

Theorem 2.5 [9]. *The following assertions are equivalent:*

(i) *A is the generator of non-degenerate, locally Lipschitz continuous integrated semigroup;*
(ii) *A satisfies the condition* (HY).

We know from [9] that under condition (HY), A is the generator of a locally Lipschitz continuous integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on X. In addition, the derivative $\{S'(t)\}_{t\geqslant 0}$ of $\{S(t)\}_{t\geqslant 0}$ generates a C_0-semigroup on $\overline{D(A)}$ such that $|S'(t)x|\leqslant\overline{M}e^{\overline{\omega}t}|x|$, for all $t\geqslant 0$ and $x\in\overline{D(A)}$.

Furthermore, let A_0 be the generator of the C_0-semigroup $\{S'(t)\}_{t\geqslant 0}$, then A_0 is the part of A on $\overline{D(A)}$ defined by

$$D(A_0)=\{x\in D(A):\ Ax\in\overline{D(A)}\},$$
$$A_0x=Ax.$$

Next, we give some general properties of the integrated semigroup $\{S(t)\}_{t\geqslant 0}$.

Proposition 2.6 [4]. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$. Then for all $x \in X$ and $t \geqslant 0$, $\int_0^t S(s)x\,ds \in D(A)$ and $S(t)x = A\int_0^t S(s)x\,ds + tx$. Moreover, for all $x \in D(A)$ and $t \geqslant 0$,*

$$S(t)x \in D(A), \qquad AS(t)x = S(t)Ax$$

and

$$S(t)x = \int_0^t S(s)Ax\,ds + tx.$$

Corollary 2.7 [4]. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$, then for all $x \in X$ and $t \geqslant 0$, $S(t)x \in \overline{D(A)}$. Moreover, for any $x \in X$, $S(\cdot)x$ is right-hand side differentiable in $t \geqslant 0$ if and only if $S(t)x \in D(A)$, and in that case we have $S'(t)x = AS(t)x + x$.*

In the sequel, we give some results for the existence of solutions of the following Cauchy problem:

$$\begin{aligned} &\frac{d}{dt}x(t) = Ax(t) + f(t), \quad t \geqslant 0, \\ &x(0) = x_0 \in X, \end{aligned} \tag{2.1}$$

where A satisfies the Hille–Yosida condition without being densely defined. Let $\{S(t)\}_{t\geqslant 0}$ be the integrated semigroup generated by A, then one has the following theorem.

Theorem 2.8 [5]. *Let $f : J \to X$ is a continuous function. Then for $x_0 \in \overline{D(A)}$, there is a unique continuous function $x : J \to X$ such that*

(i) $\int_0^t x(s)\,ds \in D(A), t \in J$;

(ii) $x(t) = x_0 + A\int_0^t x(s)\,ds + \int_0^t f(s)\,ds, t \in J$;

(iii) $\|x(t)\| \leqslant \overline{M}e^{\overline{\omega}t}[\|x_0\| + \int_0^t e^{-\overline{\omega}s}\|f(s)\|\,ds],\ t \in J$.

Moreover, x satisfies the following variation of constant formula:

$$x(t) = S'(t)x_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)\,ds, \quad t \geqslant 0. \tag{2.2}$$

Proposition 2.9 [12]. *Let $A : D(A) \subseteq X \to X$ be a linear operator satisfying the Hille–Yosida condition, $\{S(t)\}_{t\geqslant 0}$ be the integrated semigroup generated by A and $f : [0,T] \to X$, $T > 0$, be a Bochner-integrable function. Then the function $K : [0,T] \to X$ defined by $K(t) = \int_0^t S(t-s)f(s)\,ds$ is continuously differentiable on $[0,T]$, and satisfies that, for $\lambda > \overline{\omega}$ and $t \in [0,T]$,*

$$R(\lambda, A)K'(t) = \int_0^t S'(t-s)R(\lambda, A)f(s)\,ds.$$

Remark 1. Let $B_\lambda = \lambda R(\lambda, A)$, then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$.

3. MAIN RESULTS

Denote $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$. We define the following classes of functions: $PC(J, X) = \{x : J \to X\colon\ x_k \in C(J_k, X), k = 0, 1, \dots, m$ and there exist $x(t_k^+), x(t_k^-), k = 1, \dots, m$ with $x(t_k) = x(t_k^-)\}$, $PC^1(J, X) = \{x \in PC(J, X)\colon$ $x'_k \in C(J_k, X), k = 0, 1, \dots, m$ and there exist $x'(t_k^+)$, $x'(t_k^-)$, $k = 1, \dots, m$ with $x'(t_k) = x'(t_k^-)\}$, where x_k and x'_k represent the restriction of x and x' to J_k, respectively $(k = 0, \dots, m)$, and $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|$.

Obviously, $PC(J, X)$ is a Banach space with the norm $\|x\|_{PC} = \max\{\|x_k\|_{J_k}, k = 0, \dots, m\}$, and $PC^1(J, X)$ is also a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Definition 3.1. We say that $x \in PC(J, X)$ is an integral solution of Eq. (1.1) if the following assertions are true:

(i) $x(0) + g(x) = x_0$;
(ii) $\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \dots, m$;
(iii) x is continuous on J_k $(k = 0, 1, \dots, m)$;
(iv) $\int_0^t [x(s) - F(s, x(h_1(s)))]\, ds \in D(A)$ for $t \in J$;
(v) $x(t) = x_0 - g(x) - F(0, x(h_1(0))) + F(t, x(h_1(t))) + A \int_0^t [x(s) - F(s, x(h_1(s)))]\, ds + \int_0^t G(s, x(h_2(s)))\, ds + \sum_{0<t_k<t} I_k x(t_k^-) - \sum_{0<t_k<t} [F(t_k, x(h_1(t_k^+))) - F(t_k, x(h_1(t_k^-)))]$, $t \in J$.

Definition 3.2. We say that $x \in PC(J, X)$ is a strict solution of Eq. (1.1) if $x \in PC^1(J, X) \cap PC(J, D(A))$ and x satisfies Eq. (1.1).

Remark 2.

(A) It is not difficult to prove that, if x is an integral solution of Eq. (1.1) on J, then for almost all $t \in J$, $x(t) - F(t, x(h_1(t))) \in \overline{D(A)}$. In particular, $x(0) - F(0, x(h_1(0))) \in \overline{D(A)}$.

(B) If x is an integral solution of Eq. (1.1) such that $x \in PC^1(J, X)$ or $x \in PC(J, D(A))$, then x is a strict solution of Eq. (1.1).

From Theorem 2.8 and (2.2) we know that $x : J \to X$ is an integral solution of Eq. (1.1), if and only if x solves the following equations:

$$
\begin{aligned}
x(t) = {} & S'(t)\big[x_0 - g(x) - F\big(0, x\big(h_1(0)\big)\big)\big] + F\big(t, x\big(h_1(t)\big)\big) \\
& + \sum_{0<t_k<t} S'(t - t_k)\big\{I_k\big(x\big(t_k^-\big)\big) \\
& \qquad - \big[F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) - F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
& + \frac{d}{dt}\int_0^t S(t - s) G\big(s, x\big(h_2(s)\big)\big)\, ds, \quad t \in J.
\end{aligned} \tag{3.1}
$$

To obtain the existence and uniqueness of the integral solutions, we make the following hypotheses:

(H_0) The operator A satisfies the condition (HY) and generates an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on X.

(H_1) The functions F, $G: J\times X\to X$ are both continuous and Lipschitz continuous in the second variable, that is, there exist constants $\alpha_0>0$ and $\beta_0>0$ such that

$$\|F(t,x_1)-F(t,x_2)\|\leqslant \alpha_0\|x_1-x_2\|$$

and

$$\|G(t,x_1)-G(t,x_2)\|\leqslant \beta_0\|x_1-x_2\|,$$

for every $t\in J$, $x_1,x_2\in X$.

(H_2) $g\in C(PC(J,X),\overline{D(A)})$, and g satisfies that

(i) there is a constant $\gamma_0>0$ such that

$$\|g(u_1)-g(u_2)\|\leqslant \gamma_0\|u_1-u_2\|_{PC},\quad \text{for any } u_1,u_2\in PC(J,X);$$

(ii) g is a completely continuous map.

(H_3) $h_i\in C(J,J)$, $i=1,2$.

(H_4) $I_k\in C(X,X)$ and there exist constants $\alpha_k>0$, $k=1,\dots,m$, such that

$$\|I_k(x)-I_k(y)\|\leqslant \alpha_k\|x-y\|,\quad x,y\in X.$$

(H_5) The semigroup $\{S'(t)\}_{t\geqslant 0}$ is compact on $(\overline{D(A)},\|\cdot\|)$, and there is a constant $M'\geqslant 1$ such that

$$\|S'(t)\|\leqslant M',\quad \text{for all } t\in J.$$

Theorem 3.3. *Assume that the conditions* (H_0)–(H_5) *are satisfied, and*

$$M'\gamma_0+(M'+1)\alpha_0+M'\sum_{k=1}^{m}\alpha_k+2mM'\alpha_0+b\overline{M}M'\beta_0<1. \tag{3.2}$$

Let $x_0-g(x)-F(0,x(h_1(0)))\in\overline{D(A)}$ *and* $I_k(x(t_k^-))-[F(t_k,x(h_1(t_k^+)))-F(t_k,x(h_1(t_k^-)))]\in\overline{D(A)}$, *then Eq.* (1.1) *has a unique integral solution* x *on* J.

Proof. For $x\in PC(J,X)$, define Px by

$$\begin{aligned}Px(t)={}&S'(t)\big[x_0-g(x)-F\big(0,x\big(h_1(0)\big)\big)\big]+F\big(t,x\big(h_1(t)\big)\big)\\&+\sum_{0<t_k<t}S'(t-t_k)\\&\qquad\times\big\{I_k\big(x\big(t_k^-\big)\big)-\big[F\big(t_k,x\big(h_1\big(t_k^+\big)\big)\big)-F\big(t_k,x\big(h_1\big(t_k^-\big)\big)\big)\big]\big\}\\&+\frac{d}{dt}\int_0^t S(t-s)G\big(s,x\big(h_2(s)\big)\big)\,ds,\quad t\in J.\end{aligned}$$

From Proposition 2.9 and the condition (HY), we get

$$
\begin{aligned}
&\left\| B_\lambda \frac{d}{dt} \int_0^t S(t-s) G\bigl(s, x\bigl(h_2(s)\bigr)\bigr)\, ds \right\| \\
&\quad = \left\| \int_0^t S'(t-s) \lambda R(\lambda, A) G\bigl(s, x\bigl(h_2(s)\bigr)\bigr)\, ds \right\| \\
&\quad \leqslant \frac{\lambda}{\lambda - \overline{\omega}} \overline{M} M' \int_0^b \bigl\| G\bigl(s, x\bigl(h_2(s)\bigr)\bigr) \bigr\|\, ds.
\end{aligned}
$$

Letting $\lambda \to \infty$, we obtain that

$$
\left\| \frac{d}{dt} S(t-s) G\bigl(s, x\bigl(h_2(s)\bigr)\bigr)\, ds \right\| \leqslant \overline{M} M' \int_0^b \bigl\| G\bigl(s, x\bigl(h_2(s)\bigr)\bigr) \bigr\|\, ds. \tag{3.3}
$$

By the hypotheses and (3.3), we can see that for every $x_1, x_2 \in PC(J, X)$ and $t \in J$,

$$
\begin{aligned}
&\bigl\|(Px_1)(t) - (Px_2)(t)\bigr\| \\
&\quad \leqslant \bigl\| S'(t)[g(x_1) - g(x_2)] \bigr\| \\
&\qquad + \bigl\| S'(t)\bigl[F\bigl(0, x_1\bigl(h_1(0)\bigr)\bigr) - F\bigl(0, x_2\bigl(h_1(0)\bigr)\bigr)\bigr] \bigr\| \\
&\qquad + \bigl\| F\bigl(t, x_1\bigl(h_1(t)\bigr)\bigr) - F\bigl(t, x_2\bigl(h_2(t)\bigr)\bigr) \bigr\| \\
&\qquad + \sum_{0<t_k<t} \bigl\| S'(t-t_k)\bigl\{\bigl[I_k\bigl(x_1\bigl(t_k^-\bigr)\bigr) - I_k\bigl(x_2\bigl(t_k^-\bigr)\bigr)\bigr] \\
&\qquad\qquad - \bigl\{\bigl[F\bigl(t_k, x_1\bigl(h_1\bigl(t_k^+\bigr)\bigr)\bigr) - F\bigl(t_k, x_1\bigl(h_1\bigl(t_k^-\bigr)\bigr)\bigr)\bigr] \\
&\qquad\qquad - \bigl[F\bigl(t_k, x_2\bigl(h_1\bigl(t_k^+\bigr)\bigr)\bigr) - F\bigl(t_k, x_2\bigl(h_1\bigl(t_k^-\bigr)\bigr)\bigr)\bigr]\bigr\}\bigr\} \bigr\| \\
&\qquad + \overline{M} M' \int_0^b \bigl\| G\bigl(s, x_1\bigl(h_2(s)\bigr)\bigr) - G\bigl(s, x_2\bigl(h_2(s)\bigr)\bigr) \bigr\|\, ds \\
&\quad \leqslant M'\gamma_0 \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\| + \bigl(M'+1\bigr)\alpha_0 \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\| \\
&\qquad + M' \sum_{k=1}^m \alpha_k \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\| \\
&\qquad + 2mM'\alpha_0 \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\| + b\overline{M} M'\beta_0 \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\| \\
&\quad = \left[M'\gamma_0 + \bigl(M'+1\bigr)\alpha_0 + M' \sum_{k=1}^m \alpha_k + 2mM'\alpha_0 + b\overline{M} M'\beta_0 \right] \\
&\qquad \times \sup_{0 \leqslant s \leqslant b} \|x_1(s) - x_2(s)\|.
\end{aligned}
$$

Consequently, by applying condition (3.2) and the contraction mapping principle, P has a unique fixed point, therefore the nonlocal Cauchy problem (1.1) has a unique integral solution on J. The proof is completed. $\square$

To prove the existence of strict solutions, we add the following assumptions:

(H_6) $F, G \in C^1(J \times X, X)$ and their partial derivatives are locally Lipschitzian with respect to the second argument. For $t \in J$ and $\phi, \psi \in X$, there exists a constant $\beta_1 > 0$ such that

$$\begin{cases} \|D_\phi F(t,\phi) - D_\phi F(t,\psi)\| \leqslant \beta_1 \|\phi - \psi\|, \\ \|D_t F(t,\phi) - D_t F(t,\psi)\| \leqslant \beta_1 \|\phi - \psi\|, \\ \|D_\phi G(t,\phi) - D_\phi G(t,\psi)\| \leqslant \beta_1 \|\phi - \psi\|, \\ \|D_t G(t,\phi) - D_t G(t,\psi)\| \leqslant \beta_1 \|\phi - \psi\|, \end{cases}$$

where $D_t F$, $D_\phi F$, $D_t G$, $D_\phi G$ denote the derivatives with respect to t and ϕ.

(H_7) $D_x I_k(x) \in C(X, X)$, $k = 1, 2, \ldots, m$, where $D_x I_k$ denote the derivatives with respect to x.

(H_8) Functions $h_1(\cdot)$ and $h_2(\cdot)$ are continuous differentiable on J with $|h_i'(t)| \leqslant 1$ and $h_i(t) \leqslant t$ for $t \in J$ $(i = 1, 2)$, and so $h_i(0) = 0$.

Theorem 3.4. *Let x be the unique integral of Eq.* (1.1) *obtained by Theorem* 3.3. *If the hypotheses* (H_0)–(H_8) *are satisfied, then x is also a strict solution of the nonlocal Cauchy problem* (1.1) *provided that*

$$\begin{aligned} & x_0 - g(x) - F\big(0, x(0)\big) \in D(A_0), \\ & y(0) - D_t F\big(0, x(0)\big) - D_\phi F\big(0, x(0)\big) y(0) h_1'(0) \in \overline{D(A)}, \\ & \big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\big] \\ & \quad - \big[y(t_k) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\ & \qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\big] \in \overline{D(A)}, \\ & y(0) - D_t F\big(0, x(0)\big) - D_\phi F\big(0, x(0)\big) y(0) h'(0) \\ & \quad = A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big), \\ & A\big\{\big[x\big(t_k^+\big) - F\big(t, x\big(h_1\big(t_k^+\big)\big)\big)\big] - \big[x\big(t_k^-\big) - F\big(t, x\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\ & \quad = \big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\ & \qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\big] \\ & \qquad - \big[y(t_k) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\ & \qquad\quad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\big]. \end{aligned} \tag{3.4}$$

Proof. By Theorem 3.3, we know that Eq. (1.1) has a unique integral solution x which is also the unique solution of Eq. (3.1).

By Corollary 2.7, the assumption $x_0 - g(x) - F(0, x(0)) \in D(A_0)$ implies that

$$\begin{aligned} & S'(t)\big[x_0 - g(x) - F\big(0, x(0)\big)\big] \\ & \quad = S(t) A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + \big[x_0 - g(x) - F\big(0, x(0)\big)\big]. \end{aligned}$$

Then Eq. (3.1) can be rewritten as

$$
\begin{aligned}
(3.5)\qquad x(t) &= F\big(t, x\big(h_1(t)\big)\big) + S(t)A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] \\
&\quad + \big[x_0 - g(x) - F\big(0, x(0)\big)\big] \\
&\quad + \sum_{0<t_k<t} S'(t-t_k)\big\{\big[x\big(t_k^+\big) - F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[x(t_k) - F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
&\quad + \frac{d}{dt}\int_0^t S(t-s)G\big(s, x\big(h_2(s)\big)\big)\,ds, \quad t \in J.
\end{aligned}
$$

Consider the following equation:

$$
(3.6)\qquad \begin{cases}
\frac{d}{dt}\big[y(t) - D_t F\big(t, x\big(h_1(t)\big)\big) - D_\phi F\big(t, x\big(h_1(t)\big)\big)y\big(h_1(t)\big)h_1'(t)\big] \\
\quad = A\big[y(t) - D_t F\big(t, x\big(h_1(t)\big)\big) - D_\phi F\big(t, x\big(h_1(t)\big)\big)y\big(h_1(t)\big)h_1'(t)\big] \\
\qquad + D_t G\big(t, x\big(h_2(t)\big)\big) \\
\qquad + D_\phi G\big(t, x\big(h_2(t)\big)\big)y\big(h_2(t)\big)h_2'(t), \quad t \in J,\ t \neq t_k, \\
\Delta y|_{t=t_k} = D_x I_k\big(x\big(t_k^-\big)\big)y\big(t_k^-\big), \quad k = 1, \dots, m. \\
y(0) = A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big) \\
\qquad + D_t F\big(0, x(0)\big) + D_\phi F\big(0, x(0)\big)y(0)h_1'(0).
\end{cases}
$$

Then, using the same reasoning as in the proof of Theorem 3.3, one can show that Eq. (3.6) has a unique integral solution $y : J \to X$ given by

$$
\begin{aligned}
(3.7)\qquad y(t) &= S'(t)\big\{A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big)\big\} \\
&\quad + D_t F\big(t, x\big(h_1(t)\big)\big) + D_\phi F\big(t, x\big(h_1(t)\big)\big)y\big(h_1(t)\big)h_1'(t) \\
&\quad + \sum_{0<t_k<t} S'(t-t_k)\big\{\big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\
&\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big)y\big(h_1\big(t_k^+\big)\big)h_1'(t_k)\big] \\
&\qquad\qquad - \big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\
&\qquad\qquad\quad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)y\big(h_1\big(t_k^-\big)\big)h_1'(t_k)\big]\big\} \\
&\quad + \frac{d}{dt}\int_0^t S(t-s)\big[D_t G\big(s, x\big(h_2(s)\big)\big) \\
&\qquad\qquad + D_\phi G\big(s, x\big(h_2(s)\big)\big)y\big(h_2(s)\big)h_2'(s)\big]\,ds, \quad t \in J.
\end{aligned}
$$

Let $w : J \to X$ be the function defined by

$$
w(t) = x(0) + \int_0^t y(s)\,ds + \sum_{0<t_k<t} I_k\big(x\big(t_k^-\big)\big), \quad t \in J.
$$

Next, we show that $x = w$ on J. Integrating both sides of Eq. (3.7) from 0 to t, we have

$$(3.8)\qquad \int_0^t y(s)\,ds = S(t)\big\{A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big)\big\}$$

$$+ \int_0^t \big[D_t F\big(s, x\big(h_1(s)\big)\big)$$

$$+ D_\phi F\big(s, x\big(h_1(s)\big)\big) y\big(h_1(s)\big) h_1'(s)\big]\,ds$$

$$+ \sum_{0<t_k<t} S\big(t - t_k\big)$$

$$\times \big\{\big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big)$$

$$- D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\big]$$

$$- \big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)$$

$$- D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\big]\big\}$$

$$+ \int_0^t S(t-s)\big[D_t G\big(s, x\big(h_2(s)\big)\big)$$

$$+ D_\phi G\big(s, x\big(h_2(s)\big)\big) y\big(h_2(s)\big) h_2'(s)\big]\,ds.$$

On the other hand, for $t \in J$,

$$\frac{d}{dt} \int_0^t S(t-s) G\big(s, w\big(h_2(s)\big)\big)\,ds$$

$$= S(t) G\big(0, w(0)\big)$$

$$+ \int_0^t S(t-s)\big[D_t G\big(s, w\big(h_2(s)\big)\big)$$

$$+ D_\phi G\big(s, w\big(h_2(s)\big)\big) y\big(h_2(s)\big) h_2'(s)\big]\,ds.$$

Consequently,

$$(3.9)\qquad S(t) G\big(0, w(0)\big) = \frac{d}{dt} \int_0^t S(t-s) G\big(s, w\big(h_2(s)\big)\big)\,ds$$

$$- \int_0^t S(t-s)\big[D_t G\big(s, w\big(h_2(s)\big)\big)$$

$$+ D_\phi G\big(s, w\big(h_2(s)\big)\big) y\big(h_2(s)\big) h_2'(s)\big]\,ds.$$

Since

$$
\begin{aligned}
w(t) - F\big(t, w(h_1(t))\big) - \big[w(0) - F\big(0, w(0)\big)\big] \\
- \sum_{0<t_k<t} \big\{\big[w\big(t_k^+\big) - F\big(t_k, w\big(h_1\big(t_k^+\big)\big)\big)\big] \\
- \big[w\big(t_k^-\big) - F\big(t_k, w\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
= \int_0^t \frac{d}{dt}\big[w(s) - F\big(s, w(h_1(s))\big)\big]\,ds \\
= \int_0^t \big[y(s) - D_t F\big(s, w(h_1(s))\big) - D_\phi F\big(s, w(h_1(s))\big) y\big(h_1(s)\big) h_1'(s)\big]\,ds
\end{aligned}
$$

and (3.8), we get that (noting that $x(0) = w(0)$)

$$
\begin{aligned}
w(t) = F\big(t, w(h_1(t))\big) + \big[w(0) - F\big(0, w(0)\big)\big] \\
+ \int_0^t \big[y(s) - D_t F\big(s, w(h_1(s))\big) \\
- D_\phi F\big(s, w(h_1(s))\big) y\big(h_1(s)\big) h_1'(s)\big]\,ds \\
+ \sum_{0<t_k<t} \big\{\big[w\big(t_k^+\big) - F\big(t_k, w\big(h_1\big(t_k^+\big)\big)\big)\big] \\
- \big[w\big(t_k^-\big) - F\big(t_k, w\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
- \int_0^t y(s)\,ds + S(t)\big\{A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big)\big\} \\
+ \int_0^t \big[D_t F\big(s, x(h_1(s))\big) + D_\phi F\big(s, x(h_1(s))\big) y\big(h_1(s)\big) h_1'(s)\big]\,ds \\
+ \sum_{0<t_k<t} S(t - t_k)\big\{\big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\
- D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\big] \\
- \big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\
- D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\big]\big\} \\
+ \int_0^t S(t - s)\big[D_t G\big(s, x(h_2(s))\big) \\
+ D_\phi G\big(s, x(h_2(s))\big) y\big(h_2(s)\big) h_2'(s)\big]\,ds
\end{aligned}
$$

$$
\begin{aligned}
&= F\big(t, w\big(h_1(t)\big)\big) + \int_0^t \big[D_t F\big(s, x\big(h_1(s)\big)\big) - D_t F\big(s, w\big(h_1(s)\big)\big)\big]\, ds \\
&\quad + \int_0^t \big[D_\phi F\big(s, x\big(h_1(s)\big)\big) - D_\phi F\big(s, w\big(h_1(s)\big)\big)\big] y\big(h_1(s)\big) h_1'(s)\, ds \\
&\quad + S(t)\big\{A\big[x_0 - g(x) - F\big(0, x(0)\big)\big] + G\big(0, x(0)\big)\big\} \\
&\quad + \big[x_0 - g(x) - F\big(0, x(0)\big)\big] \\
&\quad + \sum_{0<t_k<t} \big\{\big[w\big(t_k^+\big) - F\big(t_k, w\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[w(t_k) - F\big(t_k, w\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
&\quad + \sum_{0<t_k<t} S(t - t_k)\big\{\big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\
&\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\big] \\
&\qquad\qquad - \big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\
&\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\big]\big\} \\
&\quad + \int_0^t S(t - s)\big[D_t G\big(s, x\big(h_2(s)\big)\big) \\
&\qquad\qquad + D_\phi G\big(s, x\big(h_2(s)\big)\big) y\big(h_2(s)\big) h_2'(s)\big]\, ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x(t) - w(t) &= F\big(t, x\big(h_1(t)\big)\big) - F\big(t, w\big(h_1(t)\big)\big) \\
&\quad + \sum_{0<t_k<t} S'(t - t_k)\big\{\big[x\big(t_k^+\big) - F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[x\big(t_k^-\big) - F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)\big]\big\} \\
&\quad + \frac{d}{dt} \int_0^t S(t - s) G\big(s, x\big(h_2(s)\big)\big)\, ds - S(t) G\big(0, x(0)\big) \\
&\quad - \int_0^t \big[D_t F\big(s, x\big(h_1(s)\big)\big) - D_t F\big(s, w\big(h_2(s)\big)\big)\big]\, ds \\
&\quad - \int_0^t \big[D_\phi F\big(s, x\big(h_1(s)\big)\big) - D_\phi F\big(s, w\big(h_1(s)\big)\big)\big] \\
&\qquad \times y\big(h_1(s)\big) h_1'(s)\, ds \\
&\quad - \sum_{0<t_k<t} \big\{\big[w\big(t_k^+\big) - F\big(t_k, w\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[w\big(t_k^-\big) - F\big(t_k, w\big(h_1\big(t_k^-\big)\big)\big)\big]\big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{0<t_k<t} S(t-t_k) \\
&\qquad \times \Big\{\Big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\
&\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\Big] \\
&\qquad\qquad - \Big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\
&\qquad\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\Big]\Big\} \\
&- \int_0^t S(t-s)\Big[D_t G\big(s, x\big(h_2(s)\big)\big) \\
&\qquad\qquad + D_\phi G\big(s, x\big(h_2(s)\big)\big) y\big(h_2(s)\big) h_2'(s)\Big]\, ds.
\end{aligned}
$$

Eq. (3.9) yields to

$$
\begin{aligned}
x(t) - w(t) = {} & F\big(t, x\big(h_1(t)\big)\big) - F\big(t, w\big(h_1(t)\big)\big) \\
&+ \sum_{0<t_k<t} S'(t-t_k)\Big\{\Big[x\big(t_k^+\big) - F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big)\Big] \\
&\qquad\qquad - \Big[x\big(t_k^-\big) - F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big)\Big]\Big\} \\
&+ \frac{d}{dt}\int_0^t S(t-s)\Big[G\big(s, x\big(h_2(s)\big)\big) - G\big(s, w\big(h_2(s)\big)\big)\Big]\, ds \\
&- \int_0^t \Big[D_t F\big(s, x\big(h_1(s)\big)\big) - D_t F\big(s, w\big(h_1(s)\big)\big)\Big]\, ds \\
&- \int_0^t \Big[D_\phi F\big(s, x\big(h_1(s)\big)\big) - D_\phi F\big(s, w\big(h_1(s)\big)\big)\Big] \\
&\qquad \times y\big(h_1(s)\big) h_1'(s)\, ds \\
&- \sum_{0<t_k<t} \Big\{\Big[w\big(t_k^+\big) - F\big(t_k, w\big(h_1\big(t_k^+\big)\big)\big)\Big] \\
&\qquad - \Big[w\big(t_k^-\big) - F\big(t_k, w\big(h_1\big(t_k^-\big)\big)\big)\Big]\Big\} \\
&- \sum_{0<t_k<t} S(t-t_k) \\
&\qquad \times \Big\{\Big[y\big(t_k^+\big) - D_t F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) \\
&\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^+\big)\big)\big) y\big(h_1\big(t_k^+\big)\big) h_1'(t_k)\Big] \\
&\qquad\qquad - \Big[y\big(t_k^-\big) - D_t F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) \\
&\qquad\qquad\qquad - D_\phi F\big(t_k, x\big(h_1\big(t_k^-\big)\big)\big) y\big(h_1\big(t_k^-\big)\big) h_1'(t_k)\Big]\Big\} \\
&+ \int_0^t S(t-s)\Big[D_t G\big(s, w\big(h_2(s)\big)\big) - D_t G\big(s, x\big(h_2(s)\big)\big)\Big]\, ds
\end{aligned}
$$

$$+\int_0^t S(t-s)\big[D_\phi G\big(s,w\big(h_2(s)\big)\big)-D_\phi G\big(s,x\big(h_2(s)\big)\big)\big]$$
$$\times\, y\big(h_2(s)\big)h_2'(s)\,ds.$$

Consequently, by Proposition 2.6 and Corollary 2.7, we deduce that

$$\begin{aligned}
\|x(t)-w(t)\| &\leqslant \alpha_0 \sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \\
&\quad + \sigma(b)\int_0^t \sup_{0\leqslant \tau\leqslant s}\|x(\tau)-w(\tau)\|\,ds \\
&\quad + \sum_{0<t_k<t}\Big\|\Big\{\big[x\big(t_k^+\big)-F\big(t_k,x\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[x\big(t_k^-\big)-F\big(t_k,x\big(h_1\big(t_k^-\big)\big)\big)\big]\Big\} \\
&\qquad\qquad - \Big\{\big[w\big(t_k^+\big)-F\big(t_k,w\big(h_1\big(t_k^+\big)\big)\big)\big] \\
&\qquad\qquad - \big[w\big(t_k^-\big)-F\big(t_k,w\big(h_1\big(t_k^-\big)\big)\big)\big]\Big\}\Big\| \\
&\leqslant \alpha_0 \sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \\
&\quad + \sigma(b)\int_0^t \sup_{0\leqslant \tau\leqslant s}\|x(\tau)-w(\tau)\|\,ds \\
&\quad + \sum_{k=1}^m \alpha_k \sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \\
&\quad + 2m\alpha_0 \sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \\
&= \Big[(2m+1)\alpha_0+\sum_{k=1}^m\alpha_k\Big]\sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \\
&\quad + \sigma(b)\int_0^t \sup_{0\leqslant \tau\leqslant s}\|x(\tau)-w(\tau)\|\,ds,
\end{aligned}$$

where

$$\sigma(b)=M'\beta_0+\beta_1+\beta_1 b_0+\beta_1 b_0+\beta_1 b_0^2,$$
$$b_0=\max\Big\{\sup_{0\leqslant s\leqslant b}\|S(s)\|,\ \sup_{0\leqslant s\leqslant b}\|y(s)\|\Big\}.$$

Hence

$$\sup_{0\leqslant s\leqslant t}\|x(s)-w(s)\| \leqslant \Big[1-(2m+1)\alpha_0-\sum_{k=1}^m\alpha_k\Big]^{-1}$$
$$\times\,\sigma(b)\int_0^t \sup_{0\leqslant \tau\leqslant s}\|x(\tau)-w(\tau)\|\,ds.$$

Using the Gronwall's lemma, we conclude that

$$\|x(t) - w(t)\| = 0 \quad \text{for } t \in J.$$

Consequently, $x(\cdot) = w(\cdot)$ on J. Therefore we conclude that $x \in PC^1(J, X)$ since w has obviously this property. Thus, by Remark 2 x is a strict solution of Eq. (1.1). The proof of Theorem 3.4 is complete. □

4. AN EXAMPLE

As applications of the obtained results of this paper, we study the following impulsive partial functional differential system with nonlocal conditions:

$$\begin{aligned}
&\frac{\partial}{\partial t}\big[z(t,x) - f\big(t, z(\sin t, x)\big)\big]\\
&\quad = \frac{\partial^2}{\partial x^2}\big[z(t,x) - f\big(t, z(\sin t, x)\big)\big]\\
&\qquad + q\big(t, z(\sin t, x)\big), \quad 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant \pi,\\
&z(t,0) = z(t,\pi) = 0,\\
&z\big(t_k^+, x\big) - z\big(t_k^-, x\big) = I_k\big(z\big(t_k^-, x\big)\big), \quad k = 1, \ldots, m,\\
&z(0,x) + \sum_{i=0}^{p} \int_0^{\pi} k_i(y,x) z(s_i, y)\, dy = z_0(x), \quad 0 \leqslant x \leqslant \pi,
\end{aligned} \tag{4.1}$$

where p is a positive integer, $0 < s_0 < s_1 < \cdots < s_p < 1$, and $0 < t_1 < t_2 < \cdots < t_m < 1$. $z_0(x) \in X = C([0,\pi])$. $k_i(\cdot,\cdot)$ are continuous functions with $k_i(y,0) = k_i(y,\pi) = 0, i = 0, \ldots, p$.

Let A be the operator defined by

$$Af = f''$$

with the domain

$$D(A) = \big\{f(\cdot) \in X\colon\ f', f'' \in X,\ f(0) = f(\pi) = 0\big\}.$$

We have $\overline{D(A)} = \{f(\cdot) \in X\colon\ f(0) = f(\pi) = 0\} \neq X$ and

$$\rho(A) \supseteq (0, +\infty), \qquad \big\|(\lambda I - A)^{-1}\big\| \leqslant \frac{1}{\lambda}, \quad \text{for } \lambda > 0.$$

This implies that A satisfies the condition (HY) on X.

It is well known that A generates an integrated semigroup $\{S(t)\}_{t \geqslant 0}$ and that $\|S'(t)\| \leqslant e^{-t}$ for $t \geqslant 0$.

Define respectively $F, G : [0,1] \times X \to X$, and $g : PC([0,1], X) \to \overline{D(A)}$ by

$$F(t,z)(x) = f\big(t, z(x)\big), \qquad G(t,z)(x) = q\big(t, z(x)\big)$$

and

$$g\big(w(t)\big) = \sum_{i=0}^{p} K_i w(t_i), \quad w \in PC([0,1], X),$$

where $K_i(z)(x) = \int_0^{\pi} k_i(y,x)z(y)\,dy$. Let $h_1(t) = h_2(t) = \sin t$.

A case where the systems (4.1) can be handled by using the classical semigroup theory is that when the function f and q are assumed to satisfy

$$f(t,0) = 0, \qquad q(t,0) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{4.2}$$

In this case, the function F and G take their values in the space $\overline{D(A)}$ and the operator A generates a strongly continuous semigroup on $\overline{D(A)}$. However, here the integrated semigroup theory allows the range of F and G to be X without the condition (4.2). Now it is easy to adapt our previous results to obtain the existence of solutions for (4.1).

We assume that:

(H_9) The functions f and q are both continuous and Lipschitz continuous in the second variable, that is, there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|f(t,z_1) - f(t,z_2)\| < k_1\|z_1 - z_2\|,$$
$$\|q(t,z_1) - q(t,z_2)\| < k_2\|z_1 - z_2\|.$$

(H_{10}) $I_k \in C(X,X)$ and there exist constants $\alpha_k > 0$, $k = 1, \ldots, m$, such that

$$\|I_k(z_1) - I_k(z_2)\| \leqslant \alpha_k\|z_1 - z_2\|, \quad z_1, z_2 \in X.$$

Then from Theorem 3.3 we know that, if

$$c + 2k_1 + \sum_{k=1}^{m} \alpha_k + 2mk_1 + k_2 < 1,$$

where $c = \max\{\|K_i\|, i = 0, 1, \ldots, p\}$, then System (4.1) admits a unique integral solution.

Moreover, we assume that

(H_{11}) Functions $f, q \in PC^1([0,1], X)$, and the derivative mappings $D_t f, D_\phi f, D_t q, D_\phi q$ all satisfy the Lipschitz condition in the second variable.

Further, suppose that $k_i \in C^2$, $k''_{i_{xx}}(y,0) = k''_{i_{xx}}(y,\pi) = 0$, (3.4) and ($H_{11}$) hold. Then x is also a strict solution of System (4.1).

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